ON DIVERGENCE FORM SECOND-ORDER PDES WITH GROWING COEFFICIENTS IN W^1_p SPACES WITHOUT WEIGHTS

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ABSTRACT. We consider second-order divergence form uniformly parabolic and elliptic PDEs with bounded and VMO_x leading coefficients and possibly linearly growing lower-order coefficients. We look for solutions which are summable to the pth power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variables.

1. INTRODUCTION

We consider divergence form uniformly parabolic and elliptic second-order PDEs with bounded and VMO_x leading coefficients and possibly linearly growing lower-order coefficients. We look for solutions which are summable to the pth power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variables. In some sense we extend the results of [17], where $p = 2$, to general $p \in (1, \infty)$. However in [17] there is no regularity assumption on the leading coefficients and there are also stochastic terms in the equations.

As in [3] one of the main motivations for studying PDEs with growing first-order coefficients is filtering theory for partially observable diffusion processes.

It is generally believed that introducing weights is the most natural setting for equations with growing coefficients. When the coefficients grow it is quite natural to consider the equations in function spaces with weights that would restrict the set of solutions in such a way that all terms in the equation will be from the same space as the free terms. The present paper seems to be the first one treating the unique solvability of these equations with growing lower-order coefficients in the usual Sobolev spaces W^1_p without weights and without imposing any *special* conditions on the relations between the coefficients or on their *derivatives*.

The theory of PDEs and stochastic PDEs in Sobolev spaces *with* weights attracted some attention in the past. We do not use weights and only

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mention a few papers about stochastic PDEs in $\mathcal{L}_p$-spaces with weights in which one can find further references: [1] (mild solutions, general p), [3], [8], [9], [10] ($p = 2$ in the four last articles).

Many more papers are devoted to the theory of *deterministic* PDEs with growing coefficients in Sobolev spaces with weights. We cite only a few of them sending the reader to the references therein again because neither do we deal with weights nor use the results of these papers. It is also worth saying that our results do not generalize the results of these papers.

In most of them the coefficients are time independent, see [2], [4], [7], [21], part of the result of which are extended in [6] to time-dependent Ornstein-Uhlenbeck operators.

It is worth noting that many issues for deterministic divergence-type equations with time independent growing coefficients in $\mathcal{L}_p$ spaces with arbitrary $p \in (1, \infty)$ *without* weights were also treated previously in the literature. This was done mostly by using the semigroup approach which excludes time dependent coefficients and makes it almost impossible to use the results in the more or less general filtering theory. We briefly mention only a few recent papers sending the reader to them for additional information.

In [19] a strongly continuous in $\mathcal{L}_p$ semigroup is constructed corresponding to elliptic operators with measurable leading coefficients and Lipschitz continuous drift coefficients. In [22] it is assumed that if, for $|x| \to \infty$, the drift coefficients grow, then the zeroth-order coefficient should grow, basically, as the square of the drift. There is also a condition on the divergence of the drift coefficient. In [23] there is no zeroth-order term and the semigroup is constructed under some assumptions one of which translates into the monotonicity of $\pm b(x) - Kx$, for a constant K, if the leading term is the Laplacian. In [5] the drift coefficient is assumed to be globally Lipschitz continuous if the zeroth-order coefficient is constant.

Some conclusions in the above cited papers are quite similar to ours but the corresponding assumptions are not as general in what concerns the regularity of the coefficients. However, these papers contain a lot of additional important information not touched upon in the present paper (in particular, it is shown in [19] that the corresponding semigroup is not analytic and in [20] that the spectrum of an elliptic operator in $\mathcal{L}_p$ *depends* on p).

The technique, we apply, originated from [18] and [13] and uses special cut-off functions whose support evolves in time in a manner adapted to the drift. As there, we do not make any regularity assumptions on the coefficients in the time variable but unlike [17], where $p = 2$, we use the results of [11] where some regularity on the coefficients in x variable is needed, like, say, the condition that the second order coefficients be in VMO uniformly with respect to the time variable.

It is worth noting that considering divergence form equations in $\mathcal{L}_p$-spaces is quite useful in the treatment of filtering problems (see, for instance, [15]) especially when the power of summability is taken large and we intend to treat this issue in a subsequent paper.

The article is organized as follows. In Section 2 we describe the problem, Section 3 contains the statements of two main results, Theorem 3.1 on an apriori estimate providing, in particular, uniqueness of solutions and Theorem 3.3 about the existence of solutions. The results about Cauchy's problem and elliptic equations are also given there. Theorem 3.1 is proved in Section 5 after we prepare the necessary tools in Section 4. Theorem 3.3 is proved in the last Section 6.

As usual when we speak of "a constant" we always mean "a finite constant".

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2. Setting of the problem

We consider the second-order operator L_t

$$L_t u_t(x) = D_i\big(a_t^{ij}(x) D_j u_t(x) + \mathfrak{b}_t^i(x) u_t(x)\big) + b_t^i(x) D_i u_t(x) - c_t(x) u_t(x),$$

acting on functions $u_t(x)$ defined on $([S,T]\cap\mathbb{R})\times\mathbb{R}^d$ (the summation convention is enforced throughout the article), where S and T are such that $-\infty \le S < T \le \infty$. Naturally,

$$D_i = \frac{\partial}{\partial x^i}$$

Our main concern is proving the unique solvability of the equation

$$\partial_t u_t = L_t u_t - \lambda u_t + D_i f_t^i + f_t^0 \quad t \in [S,T]\cap\mathbb{R}, \tag{2.1}$$

with an appropriate initial condition at $t = S$ if $S > -\infty$, where $\lambda > 0$ is a constant and $\partial_t = \partial/\partial t$. The precise assumptions on the coefficients, free terms, and initial data will be given later. First we introduce appropriate function spaces.

Denote $C_0^\infty = C_0^\infty(\mathbb{R}^d)$, $\mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^d)$, and let $W_p^1 = W_p^1(\mathbb{R}^d)$ be the Sobolev space of functions u of class $\mathcal{L}_p$, such that $Du \in \mathcal{L}_p$, where Du is the gradient of u and $1 < p < \infty$. For $-\infty \le S < T \le \infty$ define

$$\mathbb{L}_p(S,T) = \mathcal{L}_p((S,T),\mathcal{L}_p), \quad \mathbb{W}_p^1(S,T) = \mathcal{L}_p((S,T), W_p^1),$$

$$\mathbb{L}_p(T) = \mathbb{L}_p(-\infty,T), \quad \mathbb{W}_p^1(T) = \mathbb{W}_p^1(-\infty,T),$$

$$\mathbb{L}_p = \mathbb{L}_p(\infty), \quad \mathbb{W}_p^1 = \mathbb{W}_p^1(\infty).$$

Remember that the elements of $\mathbb{L}_p(S,T)$ need only belong to $\mathcal{L}_p$ on a Borel subset of (S,T) of full measure. We will always assume that these elements are defined everywhere on (S,T) at least as generalized functions on $\mathbb{R}^d$. Similar situation occurs in the case of $\mathbb{W}_p^1(S,T)$.

The following definition is most appropriate for investigating our equations if the coefficients of L are bounded.

Definition 2.1. We introduce the space $\mathcal{W}_p^1(S,T)$, which is the space of functions u_t on $[S,T]\cap\mathbb{R}$ with values in the space of generalized functions on $\mathbb{R}^d$ and having the following properties:

(i) We have $u\in\mathbb{W}_p^1(S,T)$;

(ii) There exist $f^i\in\mathbb{L}_p(S,T)$, $i=0,...,d$, such that for any $\phi\in C_0^\infty$ and finite $s,t\in[S,T]$ we have

$$(u_t,\phi)=(u_s,\phi)+\int_s^t\big((f_r^0,\phi)-(f_r^i,D_i\phi)\big)\,dr. \tag{2.2}$$

In particular, for any $\phi\in C_0^\infty$, the function (u_t,ϕ) is continuous on $[S,T]\cap\mathbb{R}$. In case that property (ii) holds, we write

$$\partial_t u_t=D_i f_t^i+f_t^0,\quad t\in[S,T]\cap\mathbb{R}.$$

Definition 2.1 allows us to introduce the spaces of initial data

Definition 2.2. Let g be a generalized function. We write $g\in W_p^{1-2/p}$ if there exists a function $v_t\in\mathcal{W}_p^1(0,1)$ such that $\partial_t v_t=\Delta v_t$, $t\in[0,1]$, and $v_0=g$. In such a case we set

$$\|g\|_{W_p^{1-2/p}}=\|v\|_{\mathbb{W}_p^1(0,1)}.$$

Following Definition 2.1 we understand equation (2.1) as the requirement that for any $\phi\in C_0^\infty$ and finite $s,t\in[S,T]$ we have

$$(u_t,\phi)=(u_s,\phi)+\int_s^t\big[(b_r^iD_iu_r-(c_r+\lambda)u_r+f_r^0,\phi)$$
$$-(a_r^{ij}D_ju_r+\mathfrak{b}_r^iu_r+f_r^i,D_i\phi)\big]\,dr. \tag{2.3}$$

Observe that at this moment it is not clear that the right-hand side makes sense. Also notice that, if the coefficients of L are bounded, then any $u\in\mathcal{W}_p^1(S,T)$ is a solution of (2.1) with appropriate free terms since if (2.2) holds, then (2.1) holds as well with

$$f_t^i-a_t^{ij}D_ju_t-\mathfrak{b}_t^iu_t,\quad i=1,...,d,\quad f_t^0+(c_t+\lambda)u_t-b_t^iD_iu_t,$$

in place of f_t^i, $i=1,...,d$, and f_t^0, respectively.

We give the definition of solution of (2.1) adopted throughout the article and which in case the coefficients of L are bounded coincides with the one obtained by applying Definition 2.1.

Definition 2.3. Let $f^j\in\mathbb{L}_p(S,T)$, $j=0,...,d$ and assume that $S>-\infty$. By a solution of (2.1) with initial condition $u_S\in W_p^{1-2/p}$ we mean a function $u\in\mathbb{W}_p^1(S,T)$ (not $\mathcal{W}_p^1(S,T)$) such that

(i) For any $\phi\in C_0^\infty$ the integral with respect to dr in (2.3) is well defined and is finite for all finite $s,t\in[S,T]$;

(ii) For any $\phi\in C_0^\infty$ equation (2.3) holds for all finite $s,t\in[S,T]$.

In case $S=-\infty$ we drop mentioning initial condition in the above lines.

3. Main results

For $\rho > 0$ denote $B_\rho(x) = \{y \in \mathbb{R}^d : |x-y| < \rho\}$, $B_\rho = B_\rho(0)$.

Assumption 3.1. (i) The functions $a^{ij}_t(x)$, $\mathfrak{b}^i_t(x)$, $b^i_t(x)$, and $c_t(x)$ are real valued and Borel measurable and $c \geq 0$.

(ii) There exists a constant $\delta > 0$ such that for all values of arguments and $\xi \in \mathbb{R}^d$

$$a^{ij}\xi^i\xi^j \geq \delta|\xi|^2, \quad |a^{ij}| \leq \delta^{-1}.$$

Also, the constant $\lambda > 0$.

(iii) For any $x \in \mathbb{R}^d$ the function

$$\int_{B_1} (|\mathfrak{b}_t(x+y)| + |b_t(x+y)| + c_t(x+y))\,dy$$

is locally integrable to the p'th power on $\mathbb{R}$, where $p' = p/(p-1)$.

Notice that the matrix $a = (a^{ij})$ need not be symmetric. Also notice that in Assumption 3.1 (iii) the ball B_1 can be replaced with any other ball without changing the set of admissible coefficients $\mathfrak{b}, b, c$.

We take and fix constants $K \geq 0, \rho_0, \rho_1 \in (0,1]$, and choose a number $q = q(d,p)$ so that

$$q > \min(d,p), \quad q > \min(d,p'), \quad q \geq \max(d,p,p'). \tag{3.1}$$

The following assumptions contain a parameter $\gamma \in (0,1]$, whose value will be specified later.

Assumption 3.2. For $\mathfrak{b} := (\mathfrak{b}^1,...,\mathfrak{b}^d)$ and $b := (b^1,...,b^d)$ and $(t,x) \in \mathbb{R}^{d+1}$ we have

$$\int_{B_{\rho_1}(x)}\int_{B_{\rho_1}(x)} |\mathfrak{b}_t(y) - \mathfrak{b}_t(z)|^q\,dydz + \int_{B_{\rho_1}(x)}\int_{B_{\rho_1}(x)} |b_t(y) - b_t(z)|^q\,dydz$$
$$+ \int_{B_{\rho_1}(x)}\int_{B_{\rho_1}(x)} |c_t(y) - c_t(z)|^q\,dydz \leq KI_{q>d} + \rho_1^d\gamma.$$

Obviously, Assumption 3.2 is satisfied if b, $\mathfrak{b}$, and c are independent of x. They also are satisfied with any $q > d$, $\gamma \in (0,1]$, and $\rho_1 = 1$ on the account of choosing K appropriately if, say,

$$|\mathfrak{b}_t(x) - \mathfrak{b}_t(y)| + |b_t(x) - b_t(y)| + |c_t(x) - c_t(y)| \leq N$$

whenever $|x-y| \leq 1$, where N is a constant. We see that Assumption 3.2 allows b, $\mathfrak{b}$, and c growing linearly in x.

Assumption 3.3. For any $\rho \in (0,\rho_0]$, $s \in \mathbb{R}$, and $i,j = 1,...,d$ we have

$$\rho^{-2d-2}\int_s^{s+\rho^2}\left(\sup_{x\in\mathbb{R}^d}\int_{B_\rho(x)}\int_{B_\rho(x)} |a^{ij}_t(y) - a^{ij}_t(z)|\,dydz\right)dt \leq \gamma. \tag{3.2}$$

Obviously, the left-hand side of (3.2) is less than

$$N(d)\sup_{t\in\mathbb{R}}\sup_{|x-y|\le 2\rho}|a^{ij}_t(x)-a^{ij}_t(y)|,$$

which implies that Assumption 3.3 is satisfied with any $\gamma\in(0,1]$ if, for instance, a is uniformly continuous in x uniformly with respect to t. Recall that if a is independent of t and for any $\gamma>0$ there is a $\rho_0>0$ such that Assumption 3.3 is satisfied, then one says that a is in VMO.

Theorem 3.1. *There exist*

$$\gamma=\gamma(d,\delta,p)\in(0,1],$$
$$N=N(d,\delta,p),\quad \lambda_0=\lambda_0(d,\delta,p,\rho_0,\rho_1,K)\ge 1$$

such that, if the above assumptions are satisfied and $\lambda\ge\lambda_0$ and u is a solution of (2.1) with zero initial data (if $S>-\infty$) and some $f^j\in\mathbb{L}_p(S,T)$, then

$$\lambda\|u\|^2_{\mathbb{L}_p(S,T)}+\|Du\|^2_{\mathbb{L}_p(S,T)}\le N\big(\sum_{i=1}^d\|f^i\|^2_{\mathbb{L}_p(S,T)}+\lambda^{-1}\|f^0\|^2_{\mathbb{L}_p(S,T)}\big). \quad (3.3)$$

Notice that the main case of Theorem 3.1 is when $S=-\infty$ because if $S>-\infty$ and $u_S=0$, then the function $u_tI_{t\ge S}$ will be a solution of our equation on $(-\infty,T]\cap\mathbb{R}$ with $f^j_t=0$ for $t<S$.

This theorem provides an apriori estimate implying uniqueness of solutions. Observe that the assumption that such a solution exists is quite nontrivial because if $\mathfrak{b}_t(x)\equiv x$, it is not true that $\mathfrak{b}u\in\mathbb{L}_p(S,T)$ for arbitrary $u\in\mathbb{W}^1_p(S,T)$.

It is also worth noting that, as can be easily seen from the proof of Theorem 3.1, one can choose a function $\gamma(d,\delta,p)$ so that it is continuous in (δ,p). The same holds for N and λ_0 from Theorem 3.1.

We have a similar result for nonzero initial data.

Theorem 3.2. *Let $S>-\infty$. In Theorem 3.1 replace the assumption that $u_S=0$ with the assumption that $u_S\in W^{1-2/p}_p$. Then its statement remains true if in the right-hand side of (3.3) we add the term*

$$N\|u_S\|^2_{W^{1-2/p}_p}.$$

Proof. Take v_t from Definition 2.2 corresponding to $g=u_S$ and set

$$\tilde u_t=\begin{cases}u_t & t\ge S,\\ (t-S+1)v_{S-t} & S\ge t\ge S-1,\\ 0 & S-1\ge t\end{cases}$$

and for $i=1,...,d$ set

$$\tilde f^i_t=\begin{cases}f^i_t & t\ge S,\\ -2(t-S+1)D_iv_{S-t} & S>t\ge S-1,\\ 0 & S-1\ge t,\end{cases}$$

$$\tilde{f}^0_t = \begin{cases} f^0_t & t \geq S, \\ [1+\lambda(t-S+1)]v_{S-t} & S > t \geq S-1, \\ 0 & S-1 \geq t. \end{cases}$$

We also modify the coefficients of L by multiplying each one of them but a^{ij}_t by $I_{t\geq S}$ and setting

$$\tilde{a}^{ij}_t = \begin{cases} a^{ij}_t & t \geq S, \\ \delta^{ij} & S > t. \end{cases}$$

Here we profit from the fact that no regularity assumption on the dependence of the coefficients on t is imposed. By denoting by $\tilde{L}$ the operator with the modified coefficients we easily see that $\tilde{u}_t$ is a solution (always in the sense of Definition 2.3) of

$$\partial_t \tilde{u}_t = \tilde{L}_t \tilde{u}_t - \lambda \tilde{u}_t + D_i \tilde{f}^i_t + \tilde{f}^0_t, \quad t \leq T.$$

By Theorem 3.1

$$\lambda \|u\|^2_{\mathbb{L}_p(S,T)} + \|Du\|^2_{\mathbb{L}_p(S,T)} \leq N\big(\sum_{i=1}^d \|\tilde{f}^i\|^2_{\mathbb{L}_p(T)} + \lambda^{-1}\|\tilde{f}^0\|^2_{\mathbb{L}_p(T)}\big),$$

where

$$\|\tilde{f}^i\|^p_{\mathbb{L}_p(T)} = \|f^i\|^p_{\mathbb{L}_p(S,T)} + \|\tilde{f}^i\|^p_{\mathbb{L}_p(S-1,S)} \leq \|f^i\|^p_{\mathbb{L}_p(S,T)} + 2^p\|D_i v\|^p_{\mathbb{L}_p(0,1)}$$
$$\leq \|f^i\|^p_{\mathbb{L}_p(S,T)} + 2^p\|u_S\|^p_{W^{1-2/p}_p},$$
$$\|\tilde{f}^0\|^p_{\mathbb{L}_p(T)} \leq \|f^0\|^p_{\mathbb{L}_p(S,T)} + N(1+\lambda^p)\|v\|^p_{\mathbb{L}_p(0,1)}$$
$$\leq \|f^0\|^p_{\mathbb{L}_p(S,T)} + N(1+\lambda^p)\|u_S\|^p_{W^{1-2/p}_p}.$$

Since $\lambda \geq \lambda_0 \geq 1$, we have $1+\lambda^p \leq 2\lambda^p$ and we get our assertion thus proving the theorem.

Here is an existence theorem.

Theorem 3.3. *Let the above assumptions be satisfied with γ taken from Theorem 3.1. Take $\lambda \geq \lambda_0$, where λ_0 is defined in Theorem 3.1. Then for any $f^j \in \mathbb{L}_p(T)$, $j = 0, ..., d$, there exists a unique solution of (2.1) with $S = -\infty$.*

It turns out that the solution, if it exists, is independent of the space in which we are looking for solutions.

Theorem 3.4. *Let $1 < p_1 \leq p_2 < \infty$ and let*

$$\gamma = \inf_{p\in[p_1,p_2]} \gamma(d,\delta,p),$$

where $\gamma(d,\delta,p)$ is taken from Theorem 3.1. Suppose that Assumptions 3.1 through 3.3 are satisfied with so defined γ and with $p = p_1$ and $p = p_2$.

(i) Let $-\infty < S < T < \infty$, $f^j \in \mathbb{L}_{p_1}(S,T) \cap \mathbb{L}_{p_2}(S,T)$, $j = 0, ..., d$, $u_S \in W^{1-2/p_1}_{p_1} \cap W^{1-2/p_2}_{p_2}$, and let $u \in \mathbb{W}^1_{p_1}(S,T) \cup \mathbb{W}^1_{p_2}(S,T)$ be a solution of (2.1). Then $u \in \mathbb{W}^1_{p_1}(S,T) \cap \mathbb{W}^1_{p_2}(S,T)$.

(ii) Let $S=-\infty, T=\infty$, $f^j\in\mathbb{L}_{p_1}\cap\mathbb{L}_{p_2}$, $j=0,...,d$, *and let* $u\in\mathbb{W}^1_{p_1}\cup\mathbb{W}^1_{p_2}$ *be a solution of* (2.1) *with*

$$\lambda\geq\sup_{p\in[p_1,p_2]}\lambda_0(d,\delta,p,\rho_0,\rho_1,K),\tag{3.4}$$

where $\lambda_0(d,\delta,p,\rho_0,\rho_1,K)$ *is taken from Theorem 3.1. Then* $u\in\mathbb{W}^1_{p_1}\cap\mathbb{W}^1_{p_2}$.

This theorem is proved in Section 6. The following theorem is about Cauchy's problem with nonzero initial data.

Theorem 3.5. *Let* $S>-\infty$ *and take a function* $u_S\in W^{1-2/p}_p$. *Let the above assumptions be satisfied with* γ *taken from Theorem 3.1. Take* $\lambda\geq\lambda_0$, *where* λ_0 *is defined in Theorem 3.1. Then for any* $f^j\in\mathbb{L}_p(S,T)$, $j=0,...,d$, *there exists a unique solution of* (2.1) *with initial value* u_S.

Proof. As in the proof of Theorem 3.2 we extend our coefficients and f^j_t for $t<S$ and then find a unique solution $\tilde{u}_t$ of

$$\partial_t\tilde{u}_t=\tilde{L}_t\tilde{u}_t-\lambda\tilde{u}_t+D_i\tilde{f}^i_t+\tilde{f}^0_t\quad t\in(-\infty,T]\cap\mathbb{R},$$

By construction $(t-S+1)v_{S-t}$ satisfies this equation for $t\leq S$, so that by uniqueness (Theorem 3.1 with S in place of T) it coincides with $\tilde{u}_t$ for $t\leq S$. In particular, $\tilde{u}_S=v_0=u_S$. Furthermore $\tilde{u}$ satisfies (2.1) since the coefficients of $\tilde{L}_t$ coincide with the corresponding coefficients of L_t for finite $t\in[S,T]$. The theorem is proved.

Remark 3.1. If both S and T are finite, then in the above theorem one can take $\lambda=0$. To show this take a large $\lambda>0$ and replace the unknown function u_t with $v_te^{\lambda t}$. This leads to an equation for v_t with the additional term $-\lambda v_t$ and the free terms multiplied by $e^{-\lambda t}$. The existence of solution v will be then equivalent to the existence of u if S and T are finite.

Remark 3.2. From the above proof and from Theorem 3.4 it follows that the solution, if it exists, is independent of p in the same sense as in Theorem 3.4.

Here is a result for elliptic equations.

Theorem 3.6. *Let the coefficients of* L_t *be independent of* t, *so that we can set* $L=L_t$ *and drop the subscript* t *elsewhere, let Assumptions 3.1 (i), (ii) be satisfied, and let* $\mathfrak{b}$, b, *and* c *be locally integrable. Then there exist*

$$\gamma=\gamma(d,\delta,p)\in(0,1],$$

$$N=N(d,\delta,p),\quad \lambda_0=\lambda_0(d,\delta,p,\rho_0,\rho_1,K)\geq1$$

such that, if Assumptions 3.2 and 3.3 are satisfied and $\lambda\geq\lambda_0$ *and* u *is a* W^1_p*-solution of*

$$Lu-\lambda u+D_if^i+f^0=0\tag{3.5}$$

in $\mathbb{R}^d$ *with some* $f^j\in\mathcal{L}_p$, $j=0,...,d$, *then*

$$\lambda\|u\|^2_{\mathcal{L}_p}+\|Du\|^2_{\mathcal{L}_p}\leq N\big(\sum_{i=1}^d\|f^i\|^2_{\mathcal{L}_p}+\lambda^{-1}\|f^0\|^2_{\mathcal{L}_p}\big).\tag{3.6}$$

Furthermore, for any $f^j \in \mathcal{L}_p$, $j = 0, ..., d$, and $\lambda \geq \lambda_0$ there exists a unique solution $u \in W^1_p$ of (3.5).

This result is obtained from the previous ones in a standard way (see, for instance, the proof of Theorem 2.1 of [13]). One of remarkable features of (3.6) is that N is independent of $\mathfrak{b}$, b, and c. It is remarkable even if they are constant, when there is no assumptions on them apart from $c \geq 0$. Another point worth noting is that if $\mathfrak{b} = b \equiv 0$, then for the solution u we have $cu \in W^{-1}_p$. However, generally it is not true that $cu \in W^{-1}_p$ for *any* $u \in W^1_p$. For instance $u(x) := (1 + |x|)^{-1} \in W^1_p$ if $p > d$, but if $c(x) = |x|$, then $(1 - \Delta)^{-1/2}(cu)(x) \to 1$ as $|x| \to \infty$ and $(1 - \Delta)^{-1/2}(cu)$ is not integrable to any power $r > 1$. Therefore generally, $(L - \lambda)W^1_p \supset W^{-1}_p$ with proper inclusion, that does not happen if the coefficients of L are bounded.

Remark 3.3. It follows, from the arguments leading to the proof of Theorem 3.6(see [13]) and from Theorem 3.4, that the solution in Theorem 3.6 is independent of p like in Theorem 3.4 if γ is chosen as in Theorem 3.4 and $\lambda \geq$ RHS of $(3.4) + 1$.

4. Differentiating compositions of generalized functions with differentiable functions

Let $\mathcal{D}$ be the space of generalized functions on $\mathbb{R}^d$. We need a formula for $u_t(x + x_t)$ where u_t behaves like a function from $\mathcal{W}^1_p$ and x_t is an $\mathbb{R}^d$-valued differentiable function. The formula is absolutely natural and probably well known. We refer the reader to [16] where such a formula is derived in a much more general setting of stochastic processes. Recall that for any $v \in \mathcal{D}$ and $\phi \in C^\infty_0$ the function $(v, \phi(\cdot - x))$ is infinitely differentiable with respect to x, so that the sup in (4.1) below is measurable.

Definition 4.1. Denote by $\mathfrak{D}(S, T)$ the set of all $\mathcal{D}$-valued functions u (written as $u_t(x)$ in a common abuse of notation) on $[S, T] \cap \mathbb{R}$ such that, for any $\phi \in C^\infty_0$, the function (u_t, ϕ) is measurable. Denote by $\mathfrak{D}^1(S, T)$ the subset of $\mathfrak{D}(S, T)$ consisting of u such that, for any $\phi \in C^\infty_0$, $R \in (0, \infty)$, and finite $t_1, t_2 \in [S, T]$ such that $t_1 < t_2$ we have

$$\int_{t_1}^{t_2} \sup_{|x| \leq R} |(u_t, \phi(\cdot - x))| \, dt < \infty. \tag{4.1}$$

Definition 4.2. Let $f, u \in \mathfrak{D}(S, T)$. We say that the equation

$$\partial_t u_t(x) = f_t(x), \quad t \in [S, T] \cap \mathbb{R}, \tag{4.2}$$

holds *in the sense of distributions* if $f \in \mathfrak{D}^1(S, T)$ and for any $\phi \in C^\infty_0$ for all finite $s, t \in [S, T]$ we have

$$(u_t, \phi) = (u_s, \phi) + \int_s^t (f_r, \phi) \, dr.$$

Let x_t be an $\mathbb{R}^d$-valued function given by

$$x_t = \int_0^t \hat{b}_s \, ds,$$

where $\hat{b}_s$ is an $\mathbb{R}^d$-valued locally integrable function on $\mathbb{R}$. Here is the formula.

Theorem 4.1. *Let $f, u \in \mathfrak{D}(S,T)$. Introduce*

$$v_t(x) = u_t(x + x_t)$$

and assume that (4.2) *holds (in the sense of distributions). Then*

$$\partial_t v_t(x) = f_t(x + x_t) + \hat{b}^i_t D_i v_t(x), \quad t \in [S,T] \cap \mathbb{R}$$

(in the sense of distributions).

Corollary 4.2. *Under the assumptions of Theorem 4.1 for any $\eta \in C_0^\infty$ we have*

$$\partial_t [u_t(x)\eta(x - x_t)] = f_t(x)\eta(x - x_t) - u_t(x)\hat{b}^i_t D_i \eta(x - x_t), \quad t \in [S,T] \cap \mathbb{R}.$$

Indeed, what we claim is that for any $\phi \in C_0^\infty$ and finite $s, t \in [S,T]$

$$((u_t\phi)(\cdot + x_t), \eta) = (u_s\phi, \eta) + \int_s^t \big(\big[f_r\phi + \hat{b}^i_r D_i(u_r\phi) \big](\cdot + x_r), \eta \big) \, dr.$$

However, to obtain this result it suffices to write down an obvious equation for $u_t\phi$, then use Theorem 4.1 and, finally, use Definition 4.2 to interpret the result.

5. Proof of Theorem 3.1

Throughout this section we suppose that Assumptions 3.1, 3.2, and 3.3 are satisfied (with a $\gamma \in (0,1]$) and start with analyzing the integral in (2.3). Recall that q was introduced before Assumption 3.2.

Lemma 5.1. *Let $1 \le r < p$ and*

$$\eta := 1 + \frac{d}{p} - \frac{d}{r} \ge 0 \tag{5.1}$$

with strict inequality if $r = 1$. Then for any $U \in \mathcal{L}_r$ and $\varepsilon > 0$ there exist $V^j \in \mathcal{L}_p$, $j = 0, 1, ..., d$, such that $U = D_i V^i + V^0$ and

$$\sum_{j=1}^d \|V^j\|_{\mathcal{L}_p} \le N(d,p,r)\varepsilon^{\eta/(1-\eta)} \|U\|_{\mathcal{L}_r}, \quad \|V^0\|_{\mathcal{L}_p} \le N(d,p,r)\varepsilon^{-1} \|U\|_{\mathcal{L}_r}. \tag{5.2}$$

In particular, for any $w \in W^1_{p'}$

$$|(U,w)| \le N(d,p,r) \|U\|_{\mathcal{L}_r} \|w\|_{W^1_{p'}}.$$

Proof. If the result is true for $\varepsilon = 1$, then for arbitrary $\varepsilon > 0$ it is easily obtained by scaling. Thus let $\varepsilon = 1$ and denote by $R_0(x)$ the kernel of $(1-\Delta)^{-1}$. For $i = 1, ..., d$ set $R_i = -D_i R_0$. One knows that $R_j(x)$ decrease exponentially fast as $|x| \to \infty$ and

$$|R_j(x)| \leq \frac{N}{|x|^{d-1}}, \quad j = 0, 1, ..., d.$$

Define

$$V^j = R_j * U, \quad j = 0, 1, ..., d.$$

If $r = 1$, one obtains (5.2) from Young's inequality since, owing to the strict inequality in (5.1) we have $p < d/(d-1)$, so that $R_j \in \mathcal{L}_p$. If $r > 1$, then for ν defined by

$$\frac{1}{p} = \frac{1}{r} - \frac{\nu}{d}$$

we have $\nu \in (0, 1]$, so that

$$|R_j(x)| \leq \frac{N}{|x|^{d-\nu}}, \quad j = 0, 1, ..., d,$$

and we obtain (5.2) from the Sobolev-Hardy-Littlewood inequality. After this it only remains to notice that in the sense of generalized functions

$$D_i V^i + V_0 = R_0 * U - \Delta R_0 * U = U.$$

The lemma is proved.

Observe that by Hölder's inequality for $r = pq/(p+q)$ ($\in [1, p)$ due to $q \geq p'$, see (3.1)) we have

$$\|hv\|_{\mathcal{L}_r} \leq \|h\|_{\mathcal{L}_q} \|v\|_{\mathcal{L}_p}.$$

Furthermore, if $r = 1$, then $q = p' > d$ (see (3.1)), $p < d/(d-1)$, and $\eta > 0$. In this way we come to the following.

Corollary 5.2. *Let $h \in \mathcal{L}_q$, $v \in \mathcal{L}_p$, and $w \in W^1_{p'}$. Then for any $\varepsilon > 0$ there exist $V^j \in \mathcal{L}_p$, $j = 0, 1, ..., d$, such that $hv = D_i V^i + V^0$ and*

$$\sum_{j=1}^{d} \|V^j\|_{\mathcal{L}_p} \leq N(d,p)\varepsilon^{(q-d)/d} \|h\|_{\mathcal{L}_q} \|v\|_{\mathcal{L}_p},$$

$$\|V^0\|_{\mathcal{L}_p} \leq N(d,p)\varepsilon^{-1} \|h\|_{\mathcal{L}_q} \|v\|_{\mathcal{L}_p}.$$

In particular,

$$|(hv, w)| \leq N(d,p) \|h\|_{\mathcal{L}_q} \|v\|_{\mathcal{L}_p} \|w\|_{W^1_{p'}}. \tag{5.3}$$

Lemma 5.3. *Let $h \in \mathcal{L}_q$ and $u \in W^1_p$. Then for any $\varepsilon > 0$ we have*

$$\|hu\|_{\mathcal{L}_p} \leq N(d,p) \|h\|_{\mathcal{L}_q} \big(\varepsilon^{(q-d)/d} \|Du\|_{\mathcal{L}_p} + \varepsilon^{-1} \|u\|_{\mathcal{L}_p}\big). \tag{5.4}$$

Proof. As above it suffices to concentrate on $\varepsilon=1$. In case $q>p$ observe that by Hölder's inequality
$$\|hu\|_{\mathcal{L}_p}\leq\|h\|_{\mathcal{L}_q}\|u\|_{\mathcal{L}_s},$$
where $s=pq/(q-p)$. After that it only remains to use embedding theorems (notice that $1-d/p\geq-d/s$ since $q\geq d$). In the remaining case $q=p$, which happens only if $p>d$ (see (3.1)). In that case the above estimate remains true if we set $s=\infty$. The lemma is proved.

Before we extract some consequences from the lemma we take a nonnegative $\xi\in C_0^\infty(B_{\rho_1})$ with unit integral and define
$$\bar{b}_s(x)=\int_{B_{\rho_1}}\xi(y)b_s(x-y)\,dy,\quad\bar{\mathfrak{b}}_s(x)=\int_{B_{\rho_1}}\xi(y)\mathfrak{b}_s(x-y)\,dy,$$
$$\bar{c}_s(x)=\int_{B_{\rho_1}}\xi(y)c_s(x-y)\,dy.\tag{5.5}$$
We may assume that $|\xi|\leq N(d)\rho_1^{-d}$.

One obtains the first assertion of the following corollary from (5.3) by observing that
$$\|I_{B_{\rho_1}(x_t)}(b_t-\bar{b}_t(x_t))\|_{\mathcal{L}_q}^q=\int_{B_{\rho_1}(x_t)}|b_t-\bar{b}_t(x_t)|^q\,dx$$
$$=\int_{B_{\rho_1}(x_t)}\Big|\int_{B_{\rho_1}(x_t)}[b_t(x)-b_t(y)]\xi(x_t-y)\,dy\Big|^q\,dx$$
$$\leq N\int_{B_{\rho_1}(x_t)}\Big|\rho_1^{-d}\int_{B_{\rho_1}(x_t)}|b_t(x)-b_t(y)|\,dy\Big|^q\,dx$$
$$\leq N\rho_1^{-d}\int_{B_{\rho_1}(x_t)}\int_{B_{\rho_1}(x_t)}|b_t(x)-b_t(y)|^q\,dy\,dx\leq N\rho_1^{-d}KI_{q>d}+N\gamma.\tag{5.6}$$
The second assertion follows from estimates like (5.6) and (5.4) where one chooses ε appropriately if $q>d$.

Corollary 5.4. *Let $u\in\mathbb{W}_p^1(S,T)$, let x_s be an $\mathbb{R}^d$-valued measurable function, and let $\eta\in C_0^\infty(B_{\rho_1})$. Set $\eta_s(x)=\eta(x-x_s)$,*
$$K_1=\sup|\eta|+\sup|D\eta|.$$
Then on (S,T)

(i) For any $w\in W_{p'}^1$ and $v\in\mathcal{L}_p$
$$(|b_s-\bar{b}_s(x_s)|\eta_s v,|w|)\leq N(d,p,K)\|\eta_s v\|_{\mathcal{L}_p}\|w\|_{W_{p'}^1};$$
(ii) We have
$$\|\eta_s|\mathfrak{b}_s-\bar{\mathfrak{b}}_s(x_s)|\,u_s\|_{\mathcal{L}_p}+\|\eta_s|c_s-\bar{c}_s(x_s)|\,u_s\|_{\mathcal{L}_p}$$
$$\leq N(d,p)\gamma^{1/q}\|\eta_s Du_s\|_{\mathcal{L}_p}+N(d,p,\gamma,\rho_1,K,K_1)\|I_{B_{\rho_1}(x_s)}u_s\|_{\mathcal{L}_p}.$$
(iii) Almost everywhere on (S,T) we have
$$(b_s^i-\bar{b}_s^i(x_s))\eta_s D_i u_s=D_iV_s^i+V_s^0,\tag{5.7}$$

$$\sum_{j=1}^{d}\|V^j\|_{\mathcal{L}_p}\leq N(d,p)\gamma^{1/q}\|\eta_s Du_s\|_{\mathcal{L}_p},$$

$$\|V^0_s\|_{\mathcal{L}_p}\leq N(d,p,\gamma,\rho_1,K)\|\eta_s Du_s\|_{\mathcal{L}_p},\tag{5.8}$$

where V^j_s, $j=0,...,d$, are some measurable $\mathcal{L}_p$-valued functions on (S,T).

To prove (iii) observe that one can find a Borel set $A\subset(S,T)$ of full measure such that $I_A D_i u$, $i=1,...,d$, are well defined as $\mathcal{L}_p$-valued Borel measurable functions. Then (5.7) with $I_A D_i u$ in place of $D_i u$ and (5.8) follow from (5.6), Corollary 5.2, and the fact that the way V^j are constructed uses bounded hence continuous operators and translates the measurability of the data into the measurability of the result. Since we are interested in (5.7) and (5.8) holding only almost everywhere on (S,T), there is no actual need for the replacement.

Corollary 5.5. *Let $u\in\mathbb{W}^1_p(S,T)$, $R\in(0,\infty)$, $\phi\in C^\infty_0(B_R)$, and let finite $S',T'\in(S,T)$ be such that $S'<T'$. Then there is a constant N independent of u and ϕ such that*

$$\int_{S'}^{T'}\big(|(b^i_sD_iu_s,\phi)|+|(\mathfrak{b}^i_su_s,D_i\phi)|+|(c_su_s,\phi)|\big)\,ds\leq N\|u\|_{\mathbb{W}^1_p(S,T)}\|\phi\|_{W^1_{p'}},\tag{5.9}$$

so that requirement (i) in Definition 2.3 can be dropped.

Proof. By having in mind partitions of unity we convince ourselves that it suffices to prove (5.9) under the assumption that ϕ has support in a ball B of radius ρ_1. Let x_0 be the center of B and set $x_s\equiv x_0$. Observe that the estimates from Corollary 5.4 imply that

$$|(\mathfrak{b}^i_su_s,D_i\phi)|\leq|(\mathfrak{b}^i_s-\bar{\mathfrak{b}}^i_s(x_0))u_s,D_i\phi)|+|\bar{\mathfrak{b}}^i_s(x_0)(u_s,D_i\phi)|$$

$$\leq N\|u_s\|_{W^1_p}\|\phi\|_{W^1_{p'}}+|\bar{\mathfrak{b}}_s(x_0)|\,\|u_s\|_{W^1_p}\|\phi\|_{W^1_{p'}}.$$

By recalling Assumption 3.1 (iii) and Hölder's inequality we get

$$\int_{S'}^{T'}|(\mathfrak{b}^i_su_s,D_i\phi)|\,ds\leq N\|u\|_{\mathbb{W}^1_p(S,T)}\|\phi\|_{W^1_{p'}}.$$

Similarly the integrals of $|(b^i_sD_iu_s,\phi)|$ and $|(c_su_s,\phi)|$ are estimated and the corollary is proved.

Since bounded linear operators are continuous we obtain the following.

Corollary 5.6. *Let $\phi\in C^\infty_0$, $T\in(0,\infty)$. Then the operators*

$$u.\to\int_0^{\cdot}(b^i_tD_iu_t,\phi)\,dt,\quad u.\to\int_0^{\cdot}(\mathfrak{b}^i_tu_t,D_i\phi)\,dt,\quad u.\to\int_0^{\cdot}(c_tu_t,\phi)\,dt$$

are continuous as operators from $\mathbb{W}^1_p(\infty)$ to $\mathcal{L}_p([-T,T])$.

This result will be used in Section 6.

Before we continue with the proof of Theorem 3.1, we notice that, if $u\in\mathcal{W}^1_p(S,T)$, then as we know (see, for instance, Theorem 2.1 of [14]), the function u_t is a continuous $\mathcal{L}_p$-valued function on $[S,T]\cap\mathbb{R}$.

Now we are ready to prove Theorem 3.1 in a particular case.

Lemma 5.7. *Let $\mathfrak{b}^i$, b^i, and c be independent of x and let $S=-\infty$. Then the assertion of Theorem 3.1 holds, naturally, with $\lambda_0=\lambda_0(d,\delta,p,\rho_0)$ (independent of ρ_1 and K).*

Proof. First let $c\equiv 0$. We want to use Theorem 4.1 to get rid of the first order terms. Observe that (2.1) reads as

$$\partial_t u_t=D_i(a^{ij}_tD_ju_t+[\mathfrak{b}^i_t+b^i_t]u_t+f^i_t)+f^0_t-\lambda u_t,\quad t\leq T. \tag{5.10}$$

Recall that from the start (see Definition 2.3) it is assumed that $u\in\mathbb{W}^1_p(T)$. Then one can find a Borel set $A\subset(-\infty,T)$ of full measure such that I_Af^j, $j=0,1,...,d$, and I_AD_iu, $i=1,...,d$, are well defined as $\mathcal{L}_p$-valued Borel functions satisfying

$$\int_{-\infty}^T I_A\big(\sum_{j=0}^d\|f^j_t\|^p_{\mathcal{L}_p}+\|Du_t\|^p_{\mathcal{L}_p}\big)\,dt<\infty.$$

Replacing f^j and D_iu in (5.10) with I_Af^j and I_AD_iu, respectively, will not affect (5.10). Similarly one can treat the term $h_t=(\mathfrak{b}^i_t+b^i_t)u_t$ for which

$$\int_{S'}^{T'}\|h_t\|_{\mathcal{L}_p}\,dt<\infty$$

for each finite $S',T'\in(-\infty,T]$, owing to Assumption 3.1 and the fact that $u\in\mathbb{L}_p(T)$.

After these replacements all terms on the right in (5.10) will be of class $\mathfrak{D}^1(-\infty,T)$ since a is bounded. This allows us to apply Theorem 4.1 and for

$$B^i_t=\int_0^t(\mathfrak{b}^i_s+b^i_s)\,ds,\quad \hat u_t(x)=u_t(x-B_t)$$

obtain that

$$\partial_t\hat u_t=D_i(\hat a^{ij}_tD_j\hat u_t)-\lambda\hat u_t+D_i\hat f^i_t+\hat f^0_t, \tag{5.11}$$

where

$$(\hat a^{ij}_t,\hat f^j_t)(x)=(a^{ij}_t,f^j_t)(x-B_t).$$

Obviously, $\hat u$ is in $\mathbb{W}^1_p(T)$ and its norm coincides with that of u. Equation (5.11) shows that $\hat u\in\mathcal{W}^1_p(T)$.

By Theorem 4.4 and Remark 2.4 of [11] there exist $\gamma=\gamma(d,\delta,p)$ and $\lambda_0=\lambda_0(d,\delta,p,\rho_0)$ such that if $\lambda\geq\lambda_0$, then

$$\|D\hat u\|_{\mathbb{L}_p(T)}+\lambda^{1/2}\|\hat u\|_{\mathbb{L}_p(T)}\leq N\big(\sum_{i=1}^d\|\hat f^i\|_{\mathbb{L}_p(T)}+\lambda^{-1/2}\|\hat f^0\|_{\mathbb{L}_p(T)}\big). \tag{5.12}$$

Actually, Theorem 4.4 of [11] is proved there only for $T=\infty$, but it is a standard fact that such an estimate implies what we need for any T (cf. the proof of Theorem 6.4.1 of [12]). Since the norms in $\mathcal{L}_p$ and W^1_p are translation invariant, (5.12) implies (3.3) and finishes the proof of the lemma in case $c\equiv 0$.

Our next step is to abandon the condition $c\equiv 0$ but assume that for an $S>-\infty$ we have $u_t=f^j_t=0$ for $t\le S$. Observe that without loss of generality we may assume that $T<\infty$. In that case introduce

$$\xi_t=\exp(\int_S^t c_s\,ds).$$

Then we have $v:=\xi u\in\mathbb{W}^1_p(T)$ and

$$\partial_t v_t=D_i(a^{ij}_t D_j v_t+[\mathfrak{b}^i_t+b^i_t]v_t+\xi_t f^i_t)+\xi_t f^0_t-\lambda v_t,\quad t\le T.$$

By the above result for all $T'\le T$

$$\int_{-\infty}^{T'}\xi^p_t\|Du_t\|^p_{\mathcal{L}_p}\,dt+\lambda^{p/2}\int_{-\infty}^{T'}\xi^p_t\|u_t\|^p_{\mathcal{L}_p}\,dt$$

$$\le N_1\sum_{i=0}^d\int_{-\infty}^{T'}\xi^p_t\|f^i_t\|^p_{\mathcal{L}_p}\,dt+N_1\lambda^{-p/2}\int_{-\infty}^{T'}\xi^p_t\|f^0_t\|^p_{\mathcal{L}_p}\,dt. \tag{5.13}$$

We multiply both part of (5.13) by $pc_{T'}\xi^{-p}_{T'}$ and integrate with respect to T' over (S,T). We use integration by parts observing that both parts vanish at $T'=S$. Then we obtain

$$\int_{-\infty}^{T}\|Du_t\|^p_{\mathcal{L}_p}\,dt+\lambda^{p/2}\int_{-\infty}^{T}\|u_t\|^p_{\mathcal{L}_p}\,dt$$

$$-\xi^{-p}_T\int_{-\infty}^{T}\xi^p_t\|Du_t\|^p_{\mathcal{L}_p}\,dt-\xi^{-p}_T\lambda^{p/2}\int_{-\infty}^{T}\xi^p_t\|u_t\|^p_{\mathcal{L}_p}\,dt$$

$$\le N_1\sum_{i=0}^d\int_{-\infty}^{T}\|f^i_t\|^p_{\mathcal{L}_p}\,dt+N_1\lambda^{-p/2}\int_{-\infty}^{T}\|f^0_t\|^p_{\mathcal{L}_p}\,dt.$$

$$-\xi^{-p}_T N_1\sum_{i=0}^d\int_{-\infty}^{T}\xi^p_t\|f^i_t\|^p_{\mathcal{L}_p}\,dt-\xi^{-p}_T N_1\lambda^{-p/2}\int_{-\infty}^{T}\xi^p_t\|f^0_t\|^p_{\mathcal{L}_p}\,dt.$$

By adding up this inequality with (5.13) with $T'=T$ multiplied by ξ^{-p}_T we obtain (3.3).

The last step is to avoid assuming that $u_t=0$ for large negative t. In that case we find a sequence $S_n\to-\infty$ such that $u_{S_n}\to 0$ in W^1_p and denote by v^n_t the unique solution of class $\mathbb{W}^1_p((0,1)\times\mathbb{R}^d)$ of the heat equation $\partial v^n_t=\Delta v^n_t$ with initial condition u_{S_n}. After that we modify u_t and the coefficients of

L_t for $t \le S_n$ as in the proof of Theorem 3.2 by taking there v^n_t and S_n in place of v_t and S, respectively. Then by the above result we obtain

$$\lambda\|u\|^2_{\mathbb{L}_p(S_n,T)} + \|Du\|^2_{\mathbb{L}_p(S_n,T)} \le N\big(\sum_{i=1}^d \|\tilde{f}^i\|^2_{\mathbb{L}_p(T)} + \lambda^{-1}\|\tilde{f}^0\|^2_{\mathbb{L}_p(T)}\big),$$

$$\le N\big(\sum_{i=1}^d \|f^i\|^2_{\mathbb{L}_p(T)} + \lambda^{-1}\|f^0\|^2_{\mathbb{L}_p(T)}\big) + N(1+\lambda^{-1})\|u_{S_n}\|^p_{W^1_p}.$$

By letting $n \to \infty$ we come to (3.3) and the lemma is proved.

Remark 5.1. In [11] the assumption corresponding to Assumption 3.3 is much weaker since in the corresponding counterpart of (3.2) there is no supremum over $x \in \mathbb{R}^d$. We need our stronger assumption because we need $a^{ij}_t(x - B_t)$ to satisfy the assumption in [11] for any function B_t.

To proceed further we need a construction. Recall that $\bar{\mathfrak{b}}$ and $\bar{b}$ are introduced in (5.5). From Lemma 4.2 of [13] and Assumption 3.2 it follows that, for $h_t = \bar{\mathfrak{b}}_t, \bar{b}_t$, it holds that $|D^n h_t| \le \kappa_n$, where $\kappa_n = \kappa_n(n,d,p,\rho_1,K) \ge 1$ and $D^n h_t$ is any derivative of h_t of order $n \ge 1$ with respect to x. By Corollary 4.3 of [13] we have $|h_t(x)| \le K(t)(1+|x|)$, where the function $K(t)$ is locally integrable with respect to t on $\mathbb{R}$. Owing to these properties, for any $(t_0,x_0) \in \mathbb{R}^{d+1}$, the equation

$$x_t = x_0 - \int_{t_0}^t (\bar{\mathfrak{b}}_s + \bar{b}_s)(x_s)\,ds, \quad t \ge t_0,$$

has a unique solution $x_t = x_{t_0,x_0,t}$.

Next, for $i = 1,2$ set $\chi^{(i)}(x)$ to be the indicator function of $B_{\rho_1/i}$ and introduce

$$\chi^{(i)}_{t_0,x_0,t}(x) = \chi^{(i)}(x - x_{t_0,x_0,t}).$$

Here is a crucial estimate.

Lemma 5.8. *Suppose that Assumptions 3.1, 3.2, and 3.3 are satisfied with a $\gamma \in (0,\gamma(d,p,\delta)]$, where $\gamma(d,p,\delta)$ is taken from Lemma 5.7. Take $(t_0,x_0) \in \mathbb{R}^{d+1}$ and assume that $t_0 < T$ and that we are given a function u which is a solution of (2.1) with $S = t_0$, with zero initial condition, some $f^j \in \mathbb{L}_p(t_0,T)$, and $\lambda \ge \lambda_0$, where $\lambda_0 = \lambda_0(d,\delta,p,\rho_0)$ is taken from Lemma 5.7. Then*

$$\lambda\|\chi^{(2)}_{t_0,x_0}u\|^2_{\mathbb{L}_p(t_0,T)} + \|\chi^{(2)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)}$$
$$\le N\sum_{i=1}^d \|\chi^{(1)}_{t_0,x_0}f^i\|^2_{\mathbb{L}_p(t_0,T)} + N\lambda^{-1}\|\chi^{(1)}_{t_0,x_0}f^0\|^2_{\mathbb{L}_p(t_0,T)}$$
$$+N\gamma^{2/q}\|\chi^{(1)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)} + N^*\lambda^{-1}\|\chi^{(1)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)}$$
$$+ N^*\|\chi^{(1)}_{t_0,x_0}u\|^2_{\mathbb{L}_p(t_0,T)} + N^*\lambda^{-1}\sum_{i=1}^d \|\chi^{(1)}_{t_0,x_0}f^i\|^2_{\mathbb{L}_p(t_0,T)}, \tag{5.14}$$

where and below in the proof by N we denote generic constants depending only on d,δ, and p and by N^* constants depending only on the same objects, γ, ρ_1, and K.

Proof. Shifting the origin allows us to assume that $t_0=0$ and $x_0=0$. With this stipulations we will drop the subscripts t_0,x_0.

Fix a $\zeta\in C_0^\infty$ with support in B_{ρ_1} and such that $\zeta=1$ on $B_{\rho_1/2}$ and $0\le\zeta\le1$. Set $x_t=x_{0,0,t}$,

$$\hat{\mathfrak{b}}_t=\bar{\mathfrak{b}}_t(x_t),\quad \hat b_t=\bar b_t(x_t),\quad \hat c_t=\bar c_t(x_t)$$

$$\eta_t(x)=\zeta(x-x_t),\quad v_t(x)=u_t(x)\eta_t(x).$$

The most important property of η_t is that

$$\partial_t\eta_t=(\hat{\mathfrak{b}}^i_t+\hat b^i_t)D_i\eta_t.$$

Also observe for the later that we may assume that

$$\chi^{(2)}_t\le\eta_t\le\chi^{(1)}_t,\quad |D\eta_t|\le N\rho_1^{-1}\chi^{(1)}_t, \tag{5.15}$$

where $\chi^{(i)}_t=\chi^{(i)}_{0,0,t}$ and $N=N(d)$.

By Corollary 4.2 (also see the argument before (5.11)) we obtain that for finite $t\in[0,T]$

$$\partial_t v_t=D_i(\eta_t a^{ij}_t D_ju_t+\mathfrak{b}^i_tv_t)-(a^{ij}_tD_ju_t+\mathfrak{b}^i_tu_t)D_i\eta_t$$
$$+b^i_t\eta_tD_iu_t-(c_t+\lambda)v_t+D_i(f^i_t\eta_t)-f^i_tD_i\eta_t+f^0_t\eta_t+(\hat{\mathfrak{b}}^i_t+\hat b^i_t)u_tD_i\eta_t.$$

We transform this further by noticing that

$$\eta_ta^{ij}_tD_ju_t=a^{ij}_tD_jv_t-a^{ij}_tu_tD_j\eta_t.$$

To deal with the term $b^i_t\eta_tD_iu_t$ we use Corollary 5.4 and find the corresponding functions V^j_t. Then simple arithmetics show that

$$\partial_tv_t=D_i\big(a^{ij}_tD_jv_t+\hat{\mathfrak{b}}^i_tv_t\big)-(\hat c_t+\lambda)v_t+\hat b^i_tD_iv_t+D_i\hat f^i_t+\hat f^0_t,$$

where

$$\hat f^0_t=f^0_t\eta_t-f^i_tD_i\eta_t-a^{ij}_t(D_ju_t)D_i\eta_t+(\hat{\mathfrak{b}}^i_t-\mathfrak{b}^i_t)u_tD_i\eta_t+V^0_t+(\hat c_t-c_t)u_t\eta_t,$$
$$\hat f^i_t=f^i_t\eta_t-a^{ij}_tu_tD_j\eta_t+(\mathfrak{b}^i_t-\hat{\mathfrak{b}}^i_t)u_t\eta_t+V^i_t,\quad i=1,..,d.$$

It we extend u_t and f^j_t as zero for $t<0$, then it will be seen from Lemma 5.7 that for $\lambda\ge\lambda_0$

$$\lambda\|v\|^2_{\mathbb{L}_p(0,T)}+\|Dv\|^2_{\mathbb{L}_p(0,T)}\le N\sum_{i=1}^d\|\hat f^i\|^2_{\mathbb{L}_p(0,T)}+N\lambda^{-1}\|\hat f^0\|^2_{\mathbb{L}_p(0,T)}. \tag{5.16}$$

Recall that here and below by N we denote generic constants depending only on d,δ, and p.

Now we start estimating the right-hand side of (5.16). First we deal with $\hat f^i_t$. Recall (5.15) and use Corollary 5.4 to get

$$\|(\mathfrak{b}^i_t-\hat{\mathfrak{b}}^i_t)u_t\eta_t\|^2_{\mathcal{L}_p}\le N\gamma^{2/q}\|\chi^{(1)}_tDu_t\|^2_{\mathcal{L}_p}+N^*\|\chi^{(1)}_tu_t\|^2_{\mathcal{L}_p} \tag{5.17}$$

(we remind the reader that by N^* we denote generic constants depending only on d, δ, p, γ, ρ_1, and K). By adding that

$$\|a^{ij}uD_j\eta\|^2_{\mathbb{L}_p(0,T)} \leq N^*\|\chi^{(1)}_{\cdot}u\|^2_{\mathbb{L}_p(0,T)},$$

we derive from (5.8) and (5.17) that

$$\sum_{i=1}^d \|\hat{f}^i\|^2_{\mathbb{L}_p(0,T)} \leq N\sum_{i=1}^d \|\chi^{(1)}_{\cdot} f^i\|^2_{\mathbb{L}_p(0,T)}$$
$$+ N\gamma^{2/q}\|\chi^{(1)}_{\cdot}Du\|^2_{\mathbb{L}_p(0,T)} + N^*\|\chi^{(1)}_{\cdot}u\|^2_{\mathbb{L}_p(0,T)}. \tag{5.18}$$

While estimating $\hat{f}^0$ we use (5.8) again and observe that we can deal with $(\hat{\mathfrak{b}}^i_t - \mathfrak{b}^i_t)u_tD_i\eta_t$ and $(c_t - \hat{c}_t)u_t\eta_t$ as in (5.17) this time without paying too much attention to the dependence of our constants on γ, ρ_1, and K and obtain that

$$\|(\hat{\mathfrak{b}}^i - \mathfrak{b}^i)uD_i\eta\|^2_{\mathbb{L}_p(0,T)} + \|(c_t - \hat{c}_t)u_t\eta_t\|^2_{\mathcal{L}_p}$$
$$\leq N^*(\|\chi^{(1)}_{\cdot}Du\|^2_{\mathbb{L}_p(0,T)} + \|\chi^{(1)}_{\cdot}u\|^2_{\mathbb{L}_p(0,T)}).$$

By estimating also roughly the remaining terms in $\hat{f}^0$ and combining this with (5.18) and (5.16), we see that the left-hand side of (5.16) is less than the right-hand side of (5.14). However,

$$|\chi^{(2)}_t Du_t| \leq |\eta_t Du_t| \leq |Dv_t| + |u_tD\eta_t| \leq |Dv_t| + N\rho_1^{-1}|u_t\chi^{(1)}_t|$$

which easily leads to (5.14). The lemma is proved.

Next, from the result giving "local" in space estimates we derive global in space estimates but for functions having, roughly speaking, small "past" support in the time variable. In the following lemma κ_1 is the number introduced before Lemma 5.8.

Lemma 5.9. *Suppose that Assumptions 3.1, 3.2, and 3.3 are satisfied with a $\gamma \in (0, \gamma(d,p,\delta)]$, where $\gamma(d,p,\delta)$ is taken from Lemma 5.7. Assume that u is a solution of (2.1) with $S = -\infty$, some $f^j \in \mathbb{L}_p(T)$, and $\lambda \geq \lambda_0$, where $\lambda_0 = \lambda_0(d,\delta,p,\rho_0)$ is taken from Lemma 5.7. Take a finite $t_0 \leq T$ and assume that $u_t = 0$ if $t \leq t_0$. Then for $I_{t_0} := I_{(t_0,T')}$, where $T' = (t_0 + \kappa_1^{-1}) \wedge T$, we have*

$$\lambda^{p/2}\|I_{t_0}u\|^p_{\mathbb{L}_p} + \|I_{t_0}Du\|^p_{\mathbb{L}_p} \leq N\sum_{i=1}^d \|I_{t_0}f^i\|^p_{\mathbb{L}_p} + N\lambda^{-p/2}\|I_{t_0}f^0\|^p_{\mathbb{L}_p}$$
$$+N\gamma^{p/q}\|I_{t_0}Du\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\|I_{t_0}Du\|^p_{\mathbb{L}_p}$$
$$+ N^*\|I_{t_0}u\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\sum_{i=1}^d \|I_{t_0}f^i\|^p_{\mathbb{L}_p}, \tag{5.19}$$

where and below in the proof by N we denote generic constants depending only on d, δ, and p and by N^* constants depending only on the same objects, γ, ρ_1, and K.

Proof. Take $x_0 \in \mathbb{R}^d$ and use the notation introduced before Lemma 5.8. By this lemma with T' in place of T we have

$$\lambda^{p/2}\|I_{t_0}\chi^{(2)}_{t_0,x_0}u\|^p_{\mathbb{L}_p} + \|I_{t_0}\chi^{(2)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p}$$

$$\leq N\sum_{i=1}^d \|I_{t_0}\chi^{(1)}_{t_0,x_0}f^i\|^p_{\mathbb{L}_p} + N\lambda^{-p/2}\|I_{t_0}\chi^{(1)}_{t_0,x_0}f^0\|^p_{\mathbb{L}_p}$$

$$+N\gamma^{p/q}\|I_{t_0}\chi^{(1)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\|I_{t_0}\chi^{(1)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p}$$

$$+ N^*\|I_{t_0}\chi^{(1)}_{t_0,x_0}u\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\sum_{i=1}^d \|I_{t_0}\chi^{(1)}_{t_0,x_0}f^i\|^p_{\mathbb{L}_p}. \tag{5.20}$$

One knows that for each $t \geq t_0$, the mapping $x_0 \to x_{t_0,x_0,t}$ is a diffeomorphism with Jacobian determinant given by

$$\left|\frac{\partial x_{t_0,x_0,t}}{\partial x_0}\right| = \exp\big(-\int_{t_0}^t \sum_{i=1}^d D_i[\bar{\mathfrak{b}}^i_s + \bar{b}^i_s](x_{t_0,x_0,s})\,ds\big).$$

By the way the constant κ_1 is introduced, we have

$$e^{-N\kappa_1(t-t_0)} \leq \left|\frac{\partial x_{t_0,x_0,t}}{\partial x_0}\right| \leq e^{N\kappa_1(t-t_0)},$$

where N depends only on d. Therefore, for any nonnegative Lebesgue measurable function $w(x)$ it holds that

$$e^{-N\kappa_1(t-t_0)}\int_{\mathbb{R}^d} w(y)\,dy \leq \int_{\mathbb{R}^d} w(x_{t_0,x_0,t})\,dx_0 \leq e^{N\kappa_1(t-t_0)}\int_{\mathbb{R}^d} w(y)\,dy.$$

In particular, since

$$\int_{\mathbb{R}^d} |\chi^{(i)}_{t_0,x_0,t}(x)|^p\,dx_0 = \int_{\mathbb{R}^d} |\chi^{(i)}(x - x_{t_0,x_0,t})|^p\,dx_0,$$

we have

$$e^{-N\kappa_1(t-t_0)} = N^*_i e^{-N\kappa_1(t-t_0)}\int_{\mathbb{R}^d} |\chi^{(i)}(x-y)|^p\,dy$$

$$\leq N^*_i\int_{\mathbb{R}^d} |\chi^{(i)}_{t_0,x_0,t}(x)|^p\,dx_0 \leq N^*_i e^{N\kappa_1(t-t_0)}\int_{\mathbb{R}^d} |\chi^{(i)}(x-y)|^p\,dy = e^{N\kappa_1(t-t_0)},$$

where $N^*_i = |B_1|^{-1}\rho_1^{-d}i^d$ and $|B_1|$ is the volume of B_1. It follows that

$$\int_{\mathbb{R}^d} |\chi^{(1)}_{t_0,x_0,t}(x)|^p\,dx_0 \leq (N^*_1)^{-1}e^{N\kappa_1(t-t_0)},$$

$$(N^*_2)^{-1}e^{-N\kappa_1(t-t_0)} \leq \int_{\mathbb{R}^d} |\chi^{(2)}_{t_0,x_0,t}(x)|^p\,dx_0.$$

Furthermore, since $u_t = 0$ if $t \leq t_0$ and $T' \leq t_0 + \kappa_1^{-1}$, in evaluating the norms in (5.20) we need not integrate with respect to t such that $\kappa_1(t-t_0) \geq 1$ or $\kappa_1(t-t_0) \leq 0$, so that for all t really involved we have

$$\int_{\mathbb{R}^d} |\chi^{(1)}_{t_0,x_0,t}(x)|^2\,dx_0 \leq (N^*_1)^{-1}e^N, \quad (N^*_2)^{-1}e^{-N} \leq \int_{\mathbb{R}^d} |\chi^{(2)}_{t_0,x_0,t}(x)|^2\,dx_0.$$

After this observation it only remains to integrate (5.20) through with respect to x_0 and use the fact that $N_1^* = 2^{-d} N_2^*$. The lemma is proved.

Proof of Theorem 3.1. Obviously we may assume that $S = -\infty$. Then first we show how to choose an appropriate $\gamma = \gamma(d, \delta, p) \in (0, 1]$. For one, we take it smaller than the one from Lemma 5.7. Then call N_0 the constant factor of $\gamma^{p/q} \|I_{t_0} Du\|_{\mathbb{L}_p}^p$ in (5.19). We know that $N_0 = N_0(d, \delta, p)$ and we choose $\gamma \in (0, 1]$ so that $N_0 \gamma^{p/q} \le 1/2$. Then under the conditions of Lemma 5.9 we have

$$\lambda^{p/2} \|I_{t_0} u\|_{\mathbb{L}_p}^p + \|I_{t_0} Du\|_{\mathbb{L}_p}^p \le N \sum_{i=1}^d \|I_{t_0} f^i\|_{\mathbb{L}_p}^p + N \lambda^{-p/2} \|I_{t_0} f^0\|_{\mathbb{L}_p}^p$$

$$+ N^* \lambda^{-p/2} \|I_{t_0} Du\|_{\mathbb{L}_p}^p + N^* \|I_{t_0} u\|_{\mathbb{L}_p}^p + N^* \lambda^{-p/2} \sum_{i=1}^d \|I_{t_0} f^i\|_{\mathbb{L}_p}^p. \qquad (5.21)$$

After γ has been fixed we recall that $\kappa_1 = \kappa_1(d, p, \rho_1, K)$ and take a $\zeta \in C_0^\infty(\mathbb{R})$ with support in $(0, \kappa_1^{-1})$ such that

$$\int_{-\infty}^{\infty} \zeta^p(t)\, dt = 1. \qquad (5.22)$$

For $s \in \mathbb{R}$ define $\zeta_t^s = \zeta(t-s)$, $u_t^s(x) = u_t(x) \zeta_t^s$. Obviously $u_t^s = 0$ if $t \le s \wedge T$. Therefore, we can apply (5.21) to u_t^s with $t_0 = s \wedge T$ observing that

$$\partial_t u_t^s = D_i(a_t^{ij} D_j u_t^s + \mathfrak{b}_t^i u_t^s) + b_t^i u_t^s - (c_t + \lambda) u_t^s + D_i(\zeta_t^s f_t^i) + \zeta_t^s f_t^0 + (\zeta_t^s)' u_t.$$

Then from (5.21) for $\lambda \ge \lambda_0$, where $\lambda_0 = \lambda_0(d, \delta, p, \rho_0)$ is taken from Lemma 5.7, we obtain

$$\lambda^{p/2} \|I_{s \wedge T} \zeta^s u\|_{\mathbb{L}_p}^p + \|I_{s \wedge T} \zeta^s Du\|_{\mathbb{L}_p}^p \le N \sum_{i=1}^d \|I_{s \wedge T} \zeta^s f^i\|_{\mathbb{L}_p}^p$$

$$+ N \lambda^{-p/2} \|I_{s \wedge T} \zeta^s f^0\|_{\mathbb{L}_p}^p + N \|I_{s \wedge T} (\zeta^s)' u\|_{\mathbb{L}_p}^p$$

$$+ N^* \lambda^{-p/2} \|I_{s \wedge T} \zeta^s Du\|_{\mathbb{L}_p}^p + N^* \|I_{s \wedge T} \zeta^s u\|_{\mathbb{L}_p}^p + N^* \lambda^{-p/2} \sum_{i=1}^d \|I_{s \wedge T} \zeta^s f^i\|_{\mathbb{L}_p}^p. \qquad (5.23)$$

We integrate through (5.23) with respect to $s \in \mathbb{R}$, observe that

$$I_{s \wedge T < t < [(s \wedge T) + \kappa_1^{-1}] \wedge T} = I_{t<T} I_{s \wedge T < t < (s \wedge T) + \kappa_1^{-1}} = I_{t<T} I_{s < t < s + \kappa_1^{-1}},$$

and that (5.22) yields

$$\int_{-\infty}^{\infty} I_{s \wedge T}(t) (\zeta_t^s)^p\, ds = \int_{-\infty}^{\infty} I_{s \wedge T < t < [(s \wedge T) + \kappa_1^{-1}] \wedge T} \zeta^p(t-s)\, ds$$

$$= I_{t<T} \int_{t - \kappa_1^{-1}}^{t} \zeta^p(t-s)\, ds = I_{t<T}.$$

We also notice that, since κ_1 depends only on d, p, ρ_1, K, we have

$$\int_{-\infty}^{\infty} |\zeta'(s)|^p \, ds = N^*.$$

Then we conclude

$$\lambda^{p/2}\|u\|^p_{\mathbb{L}_p(T)} + \|Du\|^p_{\mathbb{L}_p(T)} \leq N_1 \sum_{i=1}^d \|f^i\|^p_{\mathbb{L}_p(T)} + N_1\lambda^{-p/2}\|f^0\|^p_{\mathbb{L}_p(T)}$$

$$+N_1^*\lambda^{-p/2}\|Du\|^p_{\mathbb{L}_p(T)} + N_1^*\|u\|^p_{\mathbb{L}_p(T)} + N_1^*\lambda^{-p/2}\sum_{i=1}^d \|f^i\|^p_{\mathbb{L}_p(T)}.$$

Without losing generality we assume that $N_1 \geq 1$ and we show how to choose $\lambda_0 = \lambda_0(d, \delta, p, \rho_0, \rho_1, K) \geq 1$. Above we assumed that $\lambda \geq \lambda_0(d, \delta, p, \rho_0)$, where $\lambda_0(d, \delta, p, \rho_0)$ is taken from Lemma 5.7. Therefore, we take

$$\lambda_0 = \lambda_0(d, \delta, p, \rho_0, \rho_1, K) \geq \lambda_0(d, \delta, p, \rho_0)$$

such that $\lambda_0^{p/2} \geq 2N_1^*$. Then we obviously come to (3.3) (with $S = -\infty$). The theorem is proved.

6. Proof of Theorems 3.3 and 3.4

We need two auxiliary results.

Lemma 6.1. *For any $\tau, R \in (0, \infty)$, we have*

$$\int_{-\tau}^{\tau}\int_{B_R} (|\mathfrak{b}_s(x)|^{p'} + |b_s(x)|^{p'} + c_s^{p'}(x))\, dxds < \infty. \tag{6.1}$$

Proof. Obviously it suffices to prove (6.1) with $B_{\rho_1}(x_0)$ in place of B_R for any x_0. In that case, for instance, (notice that $q \geq p'$, see (3.1))

$$\int_{B_{\rho_1}(x_0)} |\mathfrak{b}_s(x)|^{p'}\, dx \leq N\Big(\int_{B_{\rho_1}(x_0)} |\mathfrak{b}_s(x) - \bar{\mathfrak{b}}_s(x_0)|^q \, dx\Big)^{p'/q} + N|\bar{\mathfrak{b}}_s(x_0)|^{p'}$$

According to (5.6)

$$\int_{B_{\rho_1}(x_0)} |\mathfrak{b}_s(x)|^{p'}\, dx \leq N + N|\bar{\mathfrak{b}}_s(x_0)|^{p'}$$

and in what concerns $\mathfrak{b}$ it only remains to use Assumption 3.1 (iii). Similarly, b_s and c_s are treated. The lemma is proved.

The solution of our equation will be obtained as the weak limit of the solutions of equations with cut-off coefficients. Therefore, the following result is appropriate. By the way, observe that usual way of proving the existence of solutions based on a priori estimates and the method of continuity cannot work in our setting mainly because of what is said after Theorem 3.6.

Lemma 6.2. *Let $\phi \in C_0^\infty$, $\tau \in (0,\infty)$. Let u^m, $u \in \mathbb{W}_p^1$, $m = 1,2,...$, be such that $u^m \to u$ weakly in $\mathbb{W}_p^1$. For $m = 1,2,...$ define $\chi_m(t) = (-m) \vee t \wedge m$, $\mathfrak{b}^i_{mt} = \chi_m(\mathfrak{b}^i_t)$, $b^i_{mt} = \chi_m(b^i_t)$, and $c_{mt} = \chi_m(c_t)$. Then the functions*

$$\int_0^t (b^i_{ms} D_i u^m_s, \phi)\, ds, \quad \int_0^t (\mathfrak{b}^i_{ms} u^m_s, D_i\phi)\, ds, \quad \int_0^t (c_{ms} u^m_s, \phi)\, ds \tag{6.2}$$

converge weakly in the space $\mathcal{L}_p([-\tau,\tau])$ as $m \to \infty$ to

$$\int_0^t (b^i_s D_i u_s, \phi)\, ds, \quad \int_0^t (\mathfrak{b}^i_s u_s, D_i\phi)\, ds, \quad \int_0^t (c_s u_s, \phi)\, ds, \tag{6.3}$$

respectively.

Proof. By Corollary 5.6 and by the fact that (strongly) continuous operators are weakly continuous we obtain that

$$\int_0^t (b^i_s D_i u^m_s, \phi)\, ds \to \int_0^t (b^i_s D_i u_s, \phi)\, ds$$

as $m \to \infty$ weakly in the space $\mathcal{L}_p([-\tau,\tau])$. Therefore, in what concerns the first function in (6.2), it suffices to show that

$$\int_0^t (D_i u^m_s, (b^i_s - b^i_{ms})\phi)\, ds \to 0$$

weakly in $\mathcal{L}_p([-\tau,\tau])$. In other words, it suffices to show that for any $\xi \in \mathcal{L}_{p'}([-\tau,\tau])$

$$\int_{-\tau}^{\tau} \xi_t \Big(\int_0^t (D_i u^m_s, (b^i_s - b^i_{ms})\phi)\, ds \Big)\, dt \to 0.$$

This relation is rewritten as

$$\int_{-\tau}^{\tau} (D_i u^m_s, \eta_s (b^i_s - b^i_{ms})\phi)\, ds \to 0, \tag{6.4}$$

where

$$\eta_s := \int_s^{\tau \operatorname{sgn} s} \xi_t\, dt$$

is bounded on $[-\tau,\tau]$. However, by the dominated convergence theorem and Lemma 6.1, we have $\eta_s(b^i_s - b^i_{ms})\phi \to 0$ as $m \to \infty$ strongly in $\mathbb{L}_{p'}(-\tau,\tau)$ and by assumption $Du^m \to Du$ weakly in $\mathbb{L}_p(-\tau,\tau)$. This implies (6.4). Similarly, one proves our assertion about the remaining functions in (6.2). The lemma is proved.

Proof of Theorem 3.3. Owing to Theorem 3.1 implying that the solution on $(-\infty,T] \cap \mathbb{R}$ is unique, without loss of generality we may assume that $T = \infty$. Define $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} as in Lemma 6.2 and consider equation (2.1) with $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} in place of $\mathfrak{b}_t$, b_t, and c_t, respectively. Obviously, $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} satisfy Assumption 3.2 with the same γ and K as $\mathfrak{b}_t$, b_t, and c_t do. By Theorem 3.1 and the method of continuity for $\lambda \geq \lambda_0(d,\delta,p,\rho_0,\rho_1,K)$ there exists a unique solution u^m of the modified equation on $\mathbb{R}$.

By Theorem 3.1 we also have

$$\|u^m\|_{\mathbb{L}_p} + \|Du^m\|_{\mathbb{L}_p} \le N,$$

where N is independent of m. Hence the sequence of functions u^m is bounded in the space $\mathbb{W}^1_p$ and consequently has a weak limit point $u \in \mathbb{W}^1_p$. For simplicity of presentation we assume that the whole sequence u^m converges weakly to u. Take a $\phi \in C_0^\infty$. Then by Lemma 6.2 the functions (6.2) converge to (6.3) weakly in $\mathcal{L}_p([-\tau,\tau])$ as $m \to \infty$ for any τ. Obviously, the same is true for $(u^m_t,\phi) \to (u_t,\phi)$ and the remaining terms entering the equation for u^m_t. Hence, by passing to the weak limit in the equation for u^m_t we see that for any $\phi \in C_0^\infty$ equation (2.3) holds for almost any $s,t \in \mathbb{R}$.

Now notice that, for each $t \in \mathbb{R}$, owing to Corollary 5.5 the equation

$$\begin{aligned}(\hat{u}_t,\phi) := \int_0^1 (u_s,\phi)\,ds + \int_0^1 \Big(\int_s^t \big[(b^i_r D_i u_r - (c_r+\lambda)u_r + f^0_r,\phi) \\ - (a^{ij}_r D_j u_r + \mathfrak{b}^i_r u_r + f^i_r, D_i\phi)\big]\,dr\Big)\,ds \qquad (6.5)\end{aligned}$$

defines a distribution. Furthermore, by the above for any $\phi \in C_0^\infty$ we have $(u_t,\phi) = (\hat{u}_t,\phi)$ (a.e.). A standard argument shows that for almost all $t \in \mathbb{R}$, $(u_t,\phi) = (\hat{u}_t,\phi)$ for any $\phi \in C_0^\infty$, that is $u_t = \hat{u}_t$ (a.e.) and $\hat{u}_t \in \mathbb{W}^1_p$. In particular, we see that we can replace u_r in (6.5) with $\hat{u}_r$. Finally, for any $t_1,t_2 \in \mathbb{R}$

$$(\hat{u}_{t_2},\phi) - (\hat{u}_{t_1},\phi) = \int_0^1 \Big(\int_{t_1}^{t_2} \big[(b^i_r D_i \hat{u}_r - (c_r+\lambda)\hat{u}_r + f^0_r,\phi)$$

$$-(a^{ij}_r D_j \hat{u}_r + \mathfrak{b}^i_r \hat{u}_r + f^i_r, D_i\phi)\big]\,dr\Big)\,ds = \int_{t_1}^{t_2} \big[(b^i_r D_i \hat{u}_r - (c_r+\lambda)\hat{u}_r + f^0_r,\phi)$$

$$-(a^{ij}_r D_j \hat{u}_r + \mathfrak{b}^i_r \hat{u}_r + f^i_r, D_i\phi)\big]\,dr$$

and the theorem is proved.

Proof of Theorem 3.4. (i) One reduces the general case to the one that $u_S = 0$ as in the proof of Theorem 3.2. Also, obviously, one can assume that λ is as large as we like, say satisfying (3.4), since S and T are finite. By continuing $u_t(x)$ as zero for $t \le S$ we see that we may assume that $S = \infty$. If we set $f^j_t = 0$ for $t \ge T$ and use Theorem 3.3 about the existence of solutions on $(-\infty,\infty)$ along with Theorem 3.1, which guarantees uniqueness of solutions on $(-\infty,T]$, then we see that we only need to prove assertion (ii) of the theorem.

(ii) In the above proof of Theorem 3.3 we have constructed the unique solutions of our equations as the weak limits of the solutions of equations with cut-off coefficients. Therefore, if we knew that the result is true for equations with bounded coefficients, then we would obtain it in our general case as well.

Thus it only remains to concentrate on equations with bounded coefficients. Existence an uniqueness theorems also show that it suffices to prove that, if u is the solution corresponding to $p = p_2$, then $u \in \mathbb{W}^1_{p_1}$.

Take a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta(0) = 1$, set $\zeta_t^n(x) = \zeta(t/n, x/n)$, and notice that $u_t^n := u_t\zeta_t^n$ satisfies

$$\partial_t u_t^n = L_t u_t^n - \lambda u_t^n + D_i f_{nt}^i + f_{nt}^0,$$

where

$$f_{nt}^i = f_t^i \zeta_t^n - u_t a_t^{ij} D_j \zeta_t^n, \quad i \geq 1,$$

$$f_{nt}^0 = f_t^0 \zeta_t^n - f_t^i D_i \zeta_t^n - (a_t^{ij} D_j u_t + a_t^i u_t) D_i \zeta_t^n - b_t^i u_t D_i \zeta_t^n + u_t \partial_t \zeta_t^n.$$

Since u_t^n has compact support and $p_1 \leq p_2$, it holds that $u^n \in \mathbb{W}_p^1$ for any $p \in [1, p_2]$ and by Theorem 3.1 for $p \in [p_1, p_2]$ we have

$$\|u^n\|_{\mathbb{W}_p^1} \leq N \sum_{i=0}^{d} \|f_n^i\|_{\mathbb{L}_p}. \tag{6.6}$$

One knows that

$$\|f^i\|_{\mathbb{L}_p} \leq N(\|f^i\|_{\mathbb{L}_{p_1}} + \|f^i\|_{\mathbb{L}_{p_2}}),$$

so that by Hölder's inequality

$$\|f_n^i\|_{\mathbb{L}_p} \leq N + N\|uD\zeta^n\|_{\mathbb{L}_p} \leq N + \|u\|_{\mathbb{L}_{p_2}} \|D\zeta^n\|_{\mathbb{L}_q},$$

with constants N independent of n, where

$$q = \frac{pp_2}{p_2 - p}.$$

Similar estimates are available for other terms in the right-hand side of (6.6). Since

$$\|\partial_t \zeta^n\|_{\mathbb{L}_q} + \|D\zeta^n\|_{\mathbb{L}_q} = N n^{-1+(p_2-p)(d+1)/(p_2 p)} \to 0$$

as $n \to \infty$ if

$$\frac{1}{p} - \frac{1}{p_2} < \frac{1}{d+1}, \tag{6.7}$$

estimate (6.6) implies that $u \in \mathbb{W}_p^1$.

Thus knowing that $u \in \mathbb{W}_{p_2}^1$ allowed us to conclude that $u \in \mathbb{W}_p^1$ as long as $p \in [p_1, p_2]$ and (6.7) holds. We can now replace p_2 with a smaller p and keep going in the same way each time increasing $1/p$ by the same amount until p reaches p_1. Then we get that $u \in \mathbb{W}_{p_1}^1$. The theorem is proved

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